Graph Theory
Homework 5

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Proposition 0.1 (Exercise 1). *Let $G = (V, E)$ be a graph. There exists an ordering $\{v_1, \ldots, v_n\}$ of the vertices so that the greedy algorithm (with this ordering) produces a minimal coloring.*

Proof. Let $k = \chi(G)$ and let $c : V \to [k]$ be a proper coloring. Order the vertices so that $v_i < v_j$ whenever $v_i \in c^{-1}(i)$ and $v_j \in c^{-1}(j)$ and $i < j$. If $c(v_i) = c(v_j)$, then the ordering of v_i, v_j is arbitrary.

$$c^{-1}(1) < c^{-1}(2) < \ldots < c^{-1}(k)$$

We claim that applying the greedy algorithm with this ordering produces a minimal coloring. When the algorithm colors the vertices in $c^{-1}(i)$, it may assign them a color less than i, but it will not assign them a color greater than i, since c is a proper coloring. Thus the greedy algorithm will not assign color greater than k to any vertex in $c^{-1}(i)$ for any i, hence the resulting coloring is minimal. □

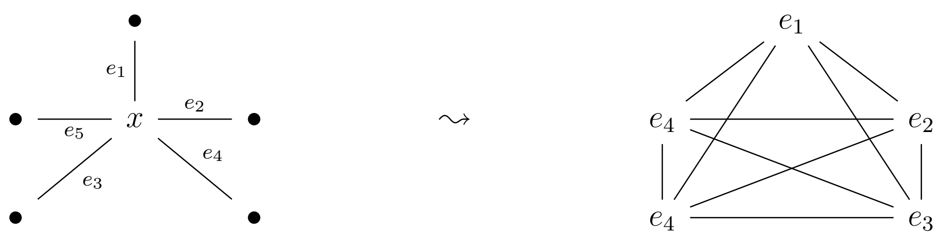
Proposition 0.2 (Exercise 2b). *Let $k \geq 3$. Then there exists a graph G_k with $\chi(G_k) = k$ but G_k does not contain K_k as a subgraph.*

Proof. Let $G_3 = C_5$ be the cycle graph with 5 vertices. Clearly $\chi(G_3) = 3$, and G_3 does not contain a 3-cycle so it contains to K_3. Now inductively define $G_{k+1} = G_k \cup \{v\}$. We also add an edge from v to each other vertex. By induction hypothesis, $\chi(G_k) = k$, and since v must be a different color from each other vertex, $\chi(G_{k+1}) = k + 1$.

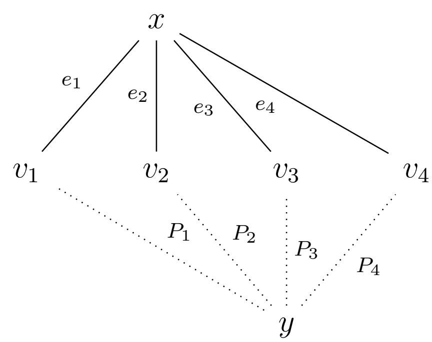
We claim that G_k does not contain a K_k subgraph. Note that G_k has 5 vertices of degree $k-1$ (the originals from G_3) and $k-3$ vertices of degree $k+1$. Thus any subset of k vertices would include at least 3 vertices from the original G_3. However, no three vertices of G_3 form a three-cycle, so any k vertex subset of G_k is missing at least one edge. □

Proposition 0.3 (Exercise 5a). *Let G be a graph such that LG is planar. Then every vertex of G has either $\deg x \leq 3$, or $\deg x = 4$ and x is a cut-vertex of G.*

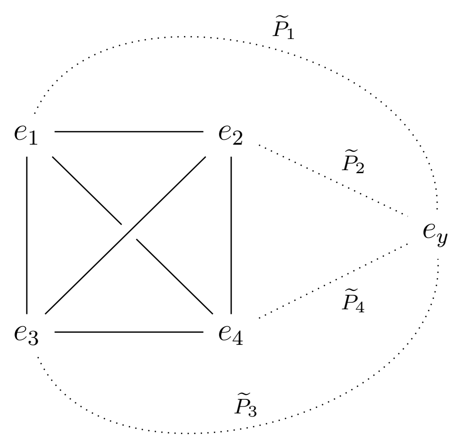
Proof. First, we show that $\deg x \leq 5$ for $x \in V(G)$. If $x \in V(G)$ has degree 5 or more, then LG has a K_5 subgraph. The picture on the left is in G, and the picture on the right is the corresponding subgraph in LG.



Suppose $\deg x = 4$ and $G \setminus x$ is connected. Let $\{v_1, v_2, v_3, v_4\} = \Gamma(x)$. Then there exists some other vertex $y \in G \setminus x$, and there are paths $v_1 P_1 y, v_2 P_2 y, v_3 P_3 y, v_4 P_4 y$ in $G \setminus v$. We depict the situation in G below.



Choose some edge e_y that has y as an endpoint. Then the path P_i in G corresponds to a path $\widetilde{P}_i$ in LG form e_i to e_y. The picture in LG corresponding to the relevant part of G is depicted below.



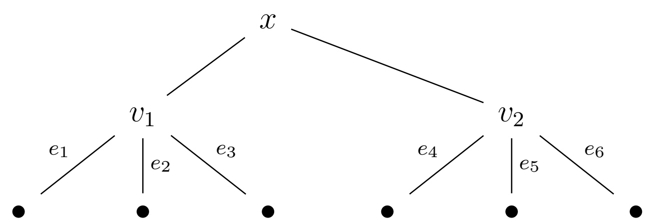
Note that the paths $\widetilde{P}_i, \widetilde{P}_j$ may overlap before reaching e_y, so this is not necessarily a TK_5 subgraph. However, even if they overlap, it is an IK_5 subgraph of LG, which contradicts LG being planar. Thus if $\deg x = 4$, then x is a cut-vertex of G. $\square$

Note: The next lemma is rather technical. The basic idea is that given two graphs G_1, G_2 with proper 4-edge-colorings and planar line graphs, we can "stitch" them together with a cutvertex of degree 4 and modify the edge colorings to get a bigger graph with a proper 4-edge-coloring.

Lemma 0.4 (Exercise 5b). *Let $G = (V, E)$ be a graph with $\Delta(G) \leq 4$ so that every vertex of degree 4 is a cutvertex. Let $x \in V$ have degree 4, and suppose that $G = G_1 \sqcup G_2$ where G_1, G_2 are disconnected by the cutvertex x, and $\chi'(G_1) \leq 4$ and $\chi'(G_2) \leq 4$. Then $\chi'(G) \leq 4$.*

Proof. Let $\Gamma(x) = \{v_1, v_2, v_3, v_4\}$. We consider two cases: (1) $v_1, v_2 \in G_1$ and $v_3, v_4 \in G_2$, and (2) $v_1, v_2, v_3 \in G_1$ and $v_4 \in G_2$. (This includes all possible cases, up to relabeling vertices.)

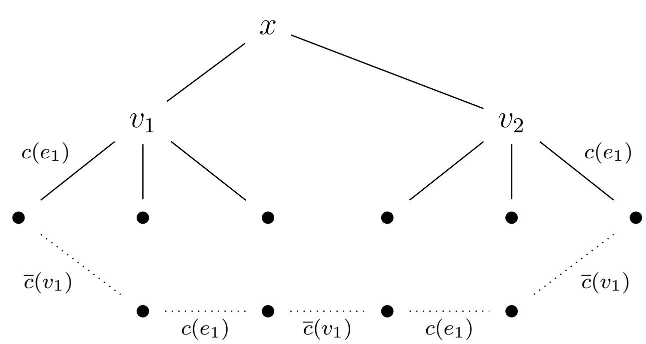
We begin with case (1), which we partially depict as below. The labellings are names, not colors, and the unlabeled vertices lie in G_1. Note that v_1, v_2 may have fewer edges into G_1 than depicted, but not more, because $\Delta(G) \leq 4$.



We have a symmetric picture for the connection between x and G_2. By assumption, we have a proper edge coloring $c : V(G_1) \to [4]$. If we can extend c to a proper edge coloring of $G \setminus G_2$, then by symmetry we can extend the coloring of G_2 to a proper edge coloring of $G \setminus G_1$. Then after permuting the labels of the edge colors for $G \setminus G_1$, we can have all four edges of x distinct, and we have a proper coloring of G. Thus to finish case (1), it suffices to extend c to a proper edge coloring of $G \setminus G_2$.

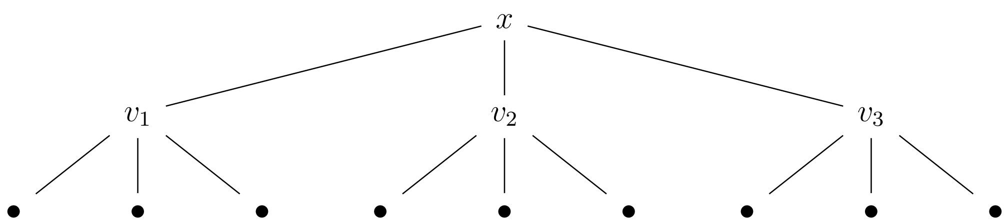
Let $\overline{c}(v_i)$ denote a color missing at vertex v_i. If $\overline{c}(v_1) \neq \overline{c}(v_2)$, then we can extend c to a proper edge coloring of $G \setminus G_2$ by setting $c(xv_i) = \overline{c}(v_i)$ for $i = 1, 2$. In particular, if v_1, v_2 do not have degree 4 in G, then we can always do this, so we may henceforth assume that $\deg_G v_1 = \deg_G v_2 = 4$.

The only remaining obstruction to extending c to $G \setminus G_2$ is if $\overline{c}(v_1) = \overline{c}(v_2)$. Then consider a "Kempe chain" of edges starting at v_1, using the alternating colors $c(e_1), \overline{c}(v_1)$. If we can interchange the colors along this chain, then we've reduced to a previous situation where $\overline{c}(v_1) \neq \overline{c}(v_2)$. The only obstruction to this interchange is if the path reaches v_2.



But if such a path exists, then v_1 is a vertex in G of degree 4 which is not a cutvertex, which is not allowed by our hypotheses. Thus no such path exists, and we may free up the color $c(e_1)$ at v_1, as desired. The proves the lemma in case (1).

Now we consider case (2). We partially depict $G \setminus G_2$ as below, where the unlabelled vertices lie in G_1.



By hypothesis, we have a proper edge coloring $c : V(G_1) \to [4]$. As in case (1), if $\overline{c}(v_1) = \overline{c}(v_2)$, then we extend c to the edges xv_1 and xv_2 by setting $c(xv_i) = \overline{c}(v_i)$ for $i = 1, 2$. If they share all colors, then we may perform a Kempe chain switch so that $\overline{c}(v_1) \neq \overline{c}(v_2)$, and then extend to xv_1, xv_2. As in case (1), the Kempe chain cannot reach v_2, so this is always possible. Now we may do the same process to ensure that $\overline{c}(v_3)$ differs from both $\overline{c}(v_1)$ and $\overline{c}(v_2)$, and complete our extension by setting $c(xv_3) = \overline{c}(v_3)$.

Finally, we consider the edge xv_4. By hypothesis we have a proper edge coloring of G_2. We know that $\deg_G v_4 \leq 4$, so there is some color missing from v_4. We simply relabel the colors for G_2 so that the missing color at v_4 is the color missing at x. This gives a proper edge coloring of all of G. This proves the lemma in case (2). □

Proposition 0.5 (Exercise 5b)**.** *Let G be a graph such that LG is planar. Then $\chi(LG) \leq 4$.*

Proof. By part (a), $\Delta(G) \leq 4$ and every vertex of degree 4 in G is a cutvertex. Let $S_G = \{v \in V(G) : \deg v = 4\}$. We induct on the size of S. For the base case, if $S = \emptyset$, then $\Delta(G) \leq 3$ and so by Vizing's Theorem, $\chi(LG) = \chi'(G) \leq 3 + 1 = 4$, and we are done.

The inductive step is essentially contained in Lemma 0.4. Let $x \in S_G$, and write $G \setminus x$ as $G_1 \sqcup G_2$ where G_1, G_2 are each unions of connected components of $G \setminus x$. Then $S_{G_1}, S_{G_2} \subset S_G$, in particular, $|S_{G_1}|, |S_{G_2}| \leq |S_G|$, so by inductive hypothesis, $\chi'(G_1), \chi'(G_2) \leq 4$. Then by the lemma, $\chi'(G) \leq 4$, which implies $\chi(LG) \leq 4$, and our induction is complete. □